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CONSEQUENCES OF NON-INTEGRABLE PERTURBATION OF INTEGRABLE CONSTRAINTS: NON-LINEAR EFFECTS OF MOTION NEAR THE EQUILIBRIUM MANIFOLD†

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A general analysis of non-linear oscillations of conservative non-holonomic systems is presented: the choice of special coordinates in a neighbourhood of the equilibrium manifold, the analytic structure of normal forms of higher approximations beginning with the second, the use of the energy integral, and the explicit form of approximate solutions.

THE EQUATIONS of motion of such systems to a first approximation have been considered both for the case of the critical point of the potential energy [1] and for the case of an arbitrary regular equilibrium [2]. The approximated constraint equations were integrated in the first paper, but not in the second. This gave rise to essentially irrelevant polemics, because in the general case it is proper to consider the neighbourhood of a manifold of equilibria rather than an isolated equilibrium [3].

Numerous investigations of the stability of non-holonomic systems (see the review [4]) have been largely based on the first approximation equation, mainly for the non-conservative case. Theorems on instability at the critical point are exceptions: the method of Chetayev functions was used [4] and asymptotic motions were constructed [5].

If all eigenvalues lie on the imaginary axis, the difference between the exact solution and the first approximation remains small only for finite times. It follows that interesting qualitative effects at long times in the motion of conservative non-holonomic systems about an equilibrium can only be found by turning to higher approximations, i.e. by utilizing the method of normal forms (see, e.g. [6, 7]). The first such investigation was the paper by Markeyev [8].

Below it is shown that non-linear oscillations of systems with non-integrable constraints can be naturally considered in the framework of the general concept of weak non-holonomicity [9], and their normal forms possess definite characteristic features.

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1. PRELIMINARY CONSIDERATIONS ON NORMALIZATION IN A NEIGHBOURHOOD OF THE EQUILIBRIUM MANIFOLD

Suppose that on the manifold (phase space) Φ there is a vector field Z that vanishes on the submanifold E. Suppose that in suitable coordinates X_k, ξ_κ E is (locally) specified by the equations $\xi_\kappa = 0$. For any function f (expressed in terms of X_k, ξ_κ) we put

$$f(X, \xi) = \sum_{M=0}^{\infty} f^{(M)}(X, \xi) = \sum_{M=0}^{\infty} \sum_{|j|=M} f^{(j)}(X)\xi^j$$

$$\frac{df}{dt} = \sum_{M=0}^{\infty} f^{[M]}(X, \xi) = \sum_{M=0}^{\infty} \sum_{|j|=M} f^{[j]}(X)\xi^j$$

$$\xi^j = \xi_1^{j_1} \cdot \xi_2^{j_2} \cdot \ldots, \quad |j| = j_1 + j_2 + \ldots$$

Because $Z = 0$ on E we have $\xi_\kappa^{[0]} = X_k^{[0]} = 0$

In order to establish a proper association with perturbation theory, we introduce into the system of differential equations a small parameter ϵ, putting $\xi = \epsilon\zeta$. Then

$$\dot{\zeta}_\kappa = \sum_{M=1}^{\infty} \epsilon^{M-1} \sum_{|j|=M} \xi_\kappa^{[j]}(X)\zeta^j, \quad \dot{X}_k = \sum_{M=1}^{\infty} \epsilon^M \sum_{|j|=M} X_k^{[j]}(X)\zeta^j$$

For $\epsilon = 0$ one obtains the original approximation, which in the theory of small oscillations is known as the first rather than the zeroth approximation. It necessarily has $\dot{X}_k^* = 0$, while ζ^* is linearly expressed in terms of ζ, with the coefficients depending on X.

We shall assume that the first-approximation system has diagonal form (so that ξ and ζ can also be complex), i.e. $\xi_\kappa^{[1]} = \lambda_\kappa(X)\xi_\kappa$. To obtain the Nth approximation, $N \geqslant 2$, one must retain terms with ϵ up to the power $N-1$ inclusive.

We consider a small ϵ-neighbourhood in the variables X, ξ. The introduction of ϵ into the equations of motion, i.e. the transition to X, ζ, means consideration of this neighbourhood in variables of order unity. The use of the Nth approximation in them gives an error of order ϵ^{N-1} over times of order $1/\epsilon$ (by the usual theorems on estimates of solutions of differential equations), i.e. the error is of order unity for $N = 1$. Consequently, an interesting theory of small oscillations (when $\operatorname{Re}\lambda_\kappa = 0$) begins at the second approximation.

It is easy to understand the appearance of the Nth approximation in the variables X, ξ. We start with a system of the form

$$\dot{X}_k = \sum_{|j|\geqslant 1} X_k^{[j]}(X)\xi^j, \quad \dot{\xi}_\kappa = \lambda_\kappa(X)\xi_\kappa + \sum_{|j|\geqslant 2} \xi_\kappa^{[j]}(X)\xi^j \tag{1.1}$$

and neglect in the right-hand sides all monomials, starting with degree N in X_k and $N+1$ in ξ_κ. One can then use a reduction-to-normal procedure, i.e. construct a change of variables after which successive approximations acquire their simplest form.

The dependence of the coefficients in system (1.1) on X introduce definite complications (which will be pointed out) compared with the normalization of ordinary quasilinear systems. We will begin with the change of variables

$$Y = X + Y^{(N-1)}(X, \xi), \quad \eta = \xi + \eta^{(N)}(X, \xi), \quad N \geqslant 2 \tag{1.2}$$

We directly quote the inverse expressions

$$X = Y - Y^{(N-1)}(Y, \eta) + \ldots, \quad \xi = \eta - \eta^{(N)}(Y, \eta) + \ldots \tag{1.3}$$

Here new symbols have simply been substituted into the expressions $Y^{(N-1)}$, $\eta^{(N)}$, and the dots indicate higher-order terms (only there do the coefficients of the polynomials (1.2) depend on X).

We differentiate relations (1.2). By (1.1) in the derivative of the monomial $F(X)\xi^j$ of degree $M \geqslant 1$ there will be monomials $(\lambda(X), j)F(X)\xi^j$ of the same degree plus terms of higher degrees including terms from the differentiation of F with respect to X. Hence in the variables X and ξ

$$Y_k^{\cdot} = X_k^{[1]} + \ldots + X_k^{[N-2]} + \sum_{|j|=N-1} (X_k^{[j]} + (\lambda, j) Y_k^{(j)}) \xi^j + \ldots$$

$$\eta_\kappa = \lambda_\kappa \xi_\kappa + \xi_k^{[2]} + \ldots + \xi_k^{[N-1]} + \sum_{|j|=N} (\xi_k^{[j]} + (\lambda, j) \eta_k^{(j)}) \xi^j + \ldots$$

On the right-hand sides of $d(1.2)/dt$ we change to variables Y, η, having substituted expressions (1.3). After this every monomial transforms as

$$F(X)\xi^j = F(Y - Y^{(N-1)} + \ldots)(\eta - \eta^{(N)} + \ldots)^j$$

This gives a term $F(Y)\eta^j$ of degree M, and then terms of degree $M+N-1$ (if one of the $j_k = 1$, and also due to the dependence of F on X), and, finally, terms of degree greater than N. The degree $N = M+N-1$ only for $M = 1$, so that the coefficients of degree N for η_κ change only as a result of the transformation of the monomial

$$\lambda_\kappa(X)\xi_\kappa = \lambda_\kappa(Y)\eta_\kappa - \sum_k \frac{\partial \lambda_\kappa}{\partial Y_k} \cdot Y_k^{(N-1)} \cdot \eta_k - \lambda_\kappa(Y)\eta^N + \ldots$$

Consequently, in variables Y, η

$$Y_k^{[j]} = X_k^{[j]} + (\lambda, j) Y_k^{(j)}, \quad |j| = N-1 \tag{1.4}$$

$$\eta_k^{[j]} = \xi_\kappa^{[j]} - \sum_k \frac{\partial \lambda_\kappa}{\partial Y_k} Y_k^{(j-e_\kappa)} + (\lambda, j - e_\kappa)\eta_\kappa^{(j)} \tag{1.5}$$

$$|j| = N, \quad j_\kappa \neq 0, \quad e_\kappa = (0, \ldots, \underset{(k)}{1}, \ldots, 0)$$

$$\eta_k^{[j]} = \xi_\kappa^{[j]} + (-\lambda_\kappa + (\lambda, j))\eta_\kappa^{(j)} \tag{1.6}$$

$$|j| = N, \quad j_\kappa = 0$$

If $(\lambda, j) \neq 0$, $|j| = N-1$, then by a suitable choice of $Y_k^{(j)}$ one can multiply $Y_k^{(j)}$, after which a choice of $\eta_\kappa^{(j)}$, $|j| = N$, $j_\kappa \neq 0$ multiplies the corresponding $\eta_\kappa^{[j]}$. One can separately multiply the $\eta_\kappa^{[j]}$, $|j| = N$, $j_\kappa = 0$ if

$$\lambda_\kappa \neq j_1\lambda_1 + \ldots + j_{\kappa-1}\lambda_{\kappa-1} + j_{\kappa+1}\lambda_{\kappa+1} + \ldots$$

When λ_κ does not depend on X, solution (1.4) does not affect (1.5), and the normalization proceeds ostensibly according to the usual scheme. However, in this case the dependence on X of other coefficients significantly lengthens the calculation of the successive approximations.

If nullification is impossible, the corresponding coefficients from (1.2) will be taken to be zero.

2. THE NEIGHBOURHOOD OF THE EQUILIBRIUM MANIFOLD IN CONSTRAINED CLASSICAL DYNAMICS

We begin by specifying linear constraints

$$g_s(x, \dot{x}) \equiv \sum_{i=1}^{n} d_{si}(x)\dot{x}_i = 0, \quad s = m+1, \ldots, n \tag{2.1}$$

and together with a Lagrangian

$$L = T - V = \tfrac{1}{2}\Sigma a_{ij}(x)\dot{x}_i\dot{x}_j - V(x)$$

The Lagrange equations with multipliers have the form (see [1])

$$d(\partial L/\partial \dot{x})/dt - \partial L/\partial x = \Sigma \mu_s \partial g_s/\partial \dot{x} \tag{2.2}$$

The equilibrium manifold is given by the conditions

$$\partial V/\partial x = \Sigma \mu_s \partial g_s/\partial x \tag{2.3}$$

In the theory it is usual to consider the case $\dim E = n-m$ (the dimension of E being equal to the number of constraints) because this is the "general situation"; the corresponding conclusions on the equations of motion can be found in [3, 4]. However, in actual problems the dimension of E is often greater. Here subtleties occur that are worth pausing over.

We shall assume the x_i to be local coordinates on the configuration manifold M. From Eqs (2.1) at each of its tangent spaces T_xM there is a distinguished subspace Π_x of dimension m (the constraint plane). We say that the submanifold E of dimension $n-l \geqslant n-m$ is properly situated with respect to the constraints if

1. T_xE is transverse to Π_x for all $x \in E$.
2. The distribution $T_xE \cap \Pi_x$ on E is completely integrable.

This distribution has dimension $m+(n-l)-n = m-l$, so that the second requirement is significant only for $l \leqslant m-2$.

We now change to more-convenient coordinates, trying not to change the notation.

Lemma 1. Without loss of generality we can assume that

$$E = \{x_\alpha = 0, \quad \alpha = 1, \ldots, l\} \tag{2.4}$$

the constraint equations near E can be represented in solved form

$$\dot{x}_s = f_s(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_m) = \sum_{\lambda=1}^{m} c_{s\lambda}(x)\dot{x}_\lambda \tag{2.5}$$

and on E the coefficients are nullified:

$$c_{s\lambda}(0, \ldots, 0, x_{l+1}, \ldots, x_n) = 0. \tag{2.6}$$

Indeed, we satisfy condition (2.4) and consider the integral manifolds E^C of the distribution $T_xE \cap \Pi_x$ on E. We have $\dim E - \dim E^C = (n-l)-(m-l) = n-m$. One can assume that on E

$$E^C = \{x_s = C_s = \text{const}; \quad s = m+1, \ldots, n\}$$

Hence the basis vectors $\partial/\partial x_s \in T_xE$ generate a plane in T_xM whose intersection with Π_x is the zero vector. We substitute $\partial/\partial x_s$ into (2.1). We obtain a matrix $\|d_{rs}\|$ which should be non-degenerate, otherwise some non-linear combinations of $\partial/\partial x_s$ would belong to Π_x, which should not happen. From this we derive expression (2.5). We introduce the expansion

$$c_{s\lambda}(x) = c^0_{s\lambda}(x_{l+1}, \ldots, x_n) + \sum_{\alpha=1}^{l} c^\alpha_{s\lambda}(x)x_\alpha$$

Because x_a $(a = l+1, \ldots, m)$ are coordinates on E^C, the vectors $\partial/\partial x_a \in T_xE \cap \Pi_x$, and hence $c^0_{sa} = 0$. If $c^0_{s\alpha} \neq 0$, then one should put $y_\lambda = x_\lambda$, $y_s = x_s - \Sigma c^0_{s\alpha}x_\alpha$, and in the new coordinates we will have

$$\dot{y}_s = \Sigma c^\alpha_{s\lambda}(x)x_\alpha\dot{x}_\lambda - \Sigma \frac{d}{dt}(c^0_{s\alpha})x_\alpha = \Sigma \tilde{c}^\beta_{s\lambda}y_\beta\dot{y}_\lambda$$

as was required.

Remarks. 1. The concluding argument is well known and has frequently been used [4]. The nature of E in Lemma 1 is not significant. This lemma remains true when $\dim E < n = m$; it is sufficient only to require that $T_xE \cap \Pi_x = 0$.

2. An example of an improperly situated equilibrium manifold is given by Chaplygin's sledge on a horizontal plane with a pendulum suspended from it. Furthermore, even for $\dim E = n-m$ the transversality of T_xE and Π_x can be violated, and then it is impossible to have (2.4) simultaneously with (2.5).

We return to conditions (2.3) and assume that they give a properly situated equilibrium manifold. We choose coordinates according to requirements (2.4)–(2.6).

Replacing (2.2) by the Voronets equations, together with (2.5) we obtain equations of the form

$$\Sigma G_{\lambda\mu}(x)\dot{v_\mu} + \Sigma\Gamma_{\lambda\mu\nu}(x)v_\mu v_\nu + \Phi_\lambda(x) = 0, \quad \dot{x_\lambda} = v_\lambda \tag{2.7}$$

where the $G_{\lambda\mu}$ are coefficients of T^* and the explicit form of $\Gamma_{\lambda\mu\nu}$ is not important here; the equilibria are now solutions of the system

$$\Phi_\lambda(x) \equiv \frac{\partial V}{\partial x_\lambda} + \Sigma \frac{\partial V}{\partial x_s} c_{s\lambda} = 0 \tag{2.8}$$

Because all the $\Phi_\lambda = 0$ when $x_\alpha = 0$, we have $\Phi_\lambda = \sum \Phi_{\lambda\beta}(x)x_\beta$ $(\beta = 1, \ldots, l)$, and on E

$$\Phi_{\lambda\beta} = \frac{\partial^2 V}{\partial x_\lambda \partial x_\beta} + \Sigma \frac{\partial V}{\partial x_s} c_{s\lambda}^{\beta} \tag{2.9}$$

We shall assume that along E

$$\det \| \Phi_{\alpha\beta} \| \neq 0 \tag{2.10}$$

Then there are clearly no other equilibria in a neighbourhood of E.

Examples with separation of variables. We take $z^\cdot = yx^\cdot$ and $L = (x^{\cdot 2} + y^{\cdot 2} + \dot{z}^{\cdot 2})/2 - W(x, z) - U(y)$ (the physical realization of this constant is discussed in [10]). The equations of motion are

$$\dot{z} = y\dot{x} \tag{2.11}$$

$$(1 + y^2)\ddot{x} + y\dot{y}\dot{x} + \partial W/\partial x + y\partial W/\partial z = 0 \tag{2.12}$$

$$\ddot{y} + \partial U/\partial y = 0 \tag{2.13}$$

It is clear that Eq. (2.13) separates. Suppose $y = 0$ is its equilibrium. Then in (2.11) we obtain $z^\cdot = 0$ (and so the constraint is integrable on the submanifold $\{y = 0\}$), and (2.12) turns into

$$\ddot{x} + \partial W(x, z)/\partial x = 0 \tag{2.14}$$

which enables us to talk of a "holonomic subproblem" with parameter z on the manifold $\{y = 0\}$. The case of an equilibrium $y = a$ is easily reduced to the preceding case.

The equilibrium equations (2.8) for system (2.11)–(2.13) have the form

$$F \equiv \partial U/\partial y = 0, \quad G \equiv \partial W/\partial x + y\partial W/\partial z = 0 \tag{2.15}$$

We construct the matrix

$$\begin{Vmatrix} \partial F/\partial x & \partial F/\partial y & \partial F/\partial z \\ \partial G/\partial x & \partial G/\partial y & \partial G/\partial z \\ -y & 0 & 1 \end{Vmatrix} \tag{2.16}$$

If its upper two rows are linearly independent, the equilibrium manifold E is one-dimensional and does not have singularities, while if it is non-degenerate, the tangent space $T_P E$ at each point $P \in E$ is transverse to the constraint plane Π_P, as is required in Lemma 1.

We substitute an arbitrary solution $y(t)$ of Eq. (2.13) into (2.11) and (2.12). Then these two equations will describe the behaviour of a system with Lagrangian $L = (x^{\cdot 2} + z^{\cdot 2})/2 - W(x, z)$ with constraint $z^\cdot = y(t)x^\cdot$. If we take a family of solutions $y = \epsilon\eta(t, \epsilon)$ near the $y = 0$ equilibrium, where ϵ is a small parameter, then we arrive at a weakly non-holonomic system in the sense of [9], and with $\epsilon = 0$ we obtain (2.14).

After the time-variable change $dt = (1 + y^2)^{-1/2}d\tau$, Eq. (2.12) acquires the form

$$x'' + \partial W/\partial x + y\partial W/\partial z = 0 \tag{2.17}$$

We shall give examples of some non-regularities in the construction of a manifold E.

An equilibrium manifold may consist of a single isolated point. We take

$$V = W + U = x^3/6 + xz^2/2 + z + y^2/2$$

Then Eqs (2.5) give

$$y = 0, \quad x^2 + z^2 = 0$$

It is worth emphasizing that the point $x = y = z = 0$ is regular both for the constraint equation and for the potential.

At points where the regular manifold E is not transverse to the constraint, there is a change of stability. We take

$$V = x^3/6 - zx + y^2/2$$

We obtain $y = 0$, $z = x^2/2$ from (2.15). There are always stable oscillations with respect to y. If however $y = 0$, then in the x, z plane we have

$$x'' = z - x^2/2, \quad z = \text{const}$$

The equilibrium $x = (2z)^{1/2}$ is stable, and $x = -(2z)^{1/2}$ is unstable. There is a change of stability at the point $x = y = z = 0$ just as the matrix (2.16) becomes degenerate. The corresponding general result follows from the proof of Lemma 2 below.

The stability mechanism for a one-dimensional equilibrium manifold deserves attention.

If one takes $U = y^2/2$, $W = \omega^2 x^2/2 + z$, then one obtains a well-known example ([11, Sec. 11]). The point was that for an equilibrium at a non-critical point of the potential (here this is $x = y = 0$ and z arbitrary) there is no analogue of the Lagrange–Dirichlet theorem, so that stability requires a special proof even if the potential energy has a minimum (here this is $z = \text{const}$) to a first approximation. With the help of the Lyapunov integral criterion a sufficient condition for stability $\omega < \pi^{-1/2}$ was obtained. This result can be extended to a necessary condition showing how instability can develop.

We take a solution of Eq. (2.13) in the form $y = \epsilon \sin t$; with respect to this variable the equilibrium $x = y = 0$ is stable; after the transition to the new time, Eq. (2.17) acquires the form of a non-homogeneous linear equation

$$x'' + \omega^2 x = -y, \quad y = \epsilon \sin t(\tau, \epsilon) \tag{2.18}$$

with the period of y with respect to τ equal to

$$\theta(\epsilon) = \int_0^{2\pi} (1 + \epsilon^2 \sin^2 t)^{-1/2} dt = (1 + \epsilon^2)^{-1/2} F(2/\pi, \epsilon(1 + \epsilon^2))^{-1/2} \tag{2.19}$$

As $\epsilon \to 0$ it is clear that $\theta(\epsilon) \to 2\pi$. Hence for stability of the $x = y = 0$ equilibrium it is sufficient to require that $\omega \neq n$.

Despite stability for fixed $\epsilon \neq 0$, Eq. (2.18) can give an unbounded solution in x when $\theta(\epsilon) = 2\pi n/\omega$. Because by (2.19) $\theta(\epsilon)$ diminishes, there is a finite number of these values of ϵ.

The instability for $\omega - n$ is obvious.

If we take $V = \omega^2 x^2/2 + xz + y^2/2$, then for Eq. (2.17) there can occur, in particular, the phenomenon of parametric resonance.

We now take the potential

$$V = -\tfrac{1}{4}\lambda \cos 2x + z + \tfrac{1}{2}y^2 \tag{2.20}$$

With an appropriate choice of coordinates and units of measurement this is the potential energy in the problem of the motion of a plate in some force field (see [10]; in particular, x is the angle of rotation of the plate). Here the holonomic subproblem is the "doubled" simple pendulum (in (2.20) we have $\cos 2x$ and not $\cos x$). Equation (2.17) acquires the form

$$x'' + \tfrac{1}{2}\lambda \cos 2x = -\epsilon \sin \tau + O(\epsilon^2)$$

Its difference from the equation of a simple pendulum perturbed by a small periodic moment is not significant, so that both general theoretical results [12, 13] and specific conclusions [14–16] apply to the problem under consideration.

We emphasize that the appearance of an equilibrium manifold in non-holonomic mechanics is not due to the non-integrability of the constraints, but to their differential representation: if $c_{s\lambda} = 0$ in (2.5), the remarks on the structure of E remain true.

From the point of view expressed in Sec. 1, the role of ξ_κ is played by x_α, v_α, v_a, and the role of X_k is that of x_a, x_s. We put $(\cdot) = (0, x_a, x_s)$. The equations of the first approximation are as follows:

$$\Sigma G_{\lambda\mu}(\cdot)\dot{v}_\mu + \Sigma \Phi_{\lambda\alpha}(\cdot) x_\alpha = 0, \quad \dot{x}_\alpha = v_\alpha, \quad \dot{x}_a = \dot{x}_r = 0 \tag{2.21}$$

We note that the $x_a^* = 0$, although the v_a participate in (2.21).

Lemma 2 (standard). We introduce a matrix $\|F_{\lambda\beta}\| = \|G_{\lambda\mu}\|^{-1}\|\Phi_{\lambda\alpha}\|$ (in general rectangular) and assume that the eigenvalues of its square part $\|F_{\alpha\beta}\|$ are positive and distinct (and if they coincide, they have simple Jordan cells). We denote them by $\omega_1^2, \ldots, \omega_l^2$. Then without loss of generality one can assume that in (2.21)

$$\|G_{\lambda\mu}\| = E^m, \quad \|\Phi_{\alpha\beta}\| = \operatorname{diag}(\omega_1^2, \ldots, \omega_l^2), \quad \|\Phi_{a\beta}\| = 0$$

so that in the variables $X = (x_a, a_r)$, $\xi = (p_{-\alpha}, p_{+\alpha}, v_a)$, where

$$p_{-\alpha} = x_\alpha - iv_\alpha/\omega_\alpha(X), \quad p_{+\alpha} = x_\alpha + iv_\alpha/\omega_\alpha(X) \equiv \overline{p_{-\alpha}}$$

The equations of the second approximation are as follows:

$$\dot{p}_{-\alpha} = i\omega_\alpha(X)p_{-\alpha} + p_{-\alpha}^{[2]}(X, \xi) \tag{2.22}$$

$$\dot{v}_a = v_a^{[2]}(X, \xi), \quad \dot{x}_a = v_a, \quad \dot{x}_r = 0$$

The last equation is true by Lemma 1.

3. THE GENERAL STRUCTURE OF THE NORMAL FORM

System (2.5), (2.7) is reversible in the sense that if $x(t)$, $v(t)$ is a solution, then so is $x(-t)$, $v(-t)$. Furthermore, it possesses a first integral, namely the energy integral quadratic in the velocities:

$$H = T^* + V = \tfrac{1}{2}\Sigma G_{\lambda\mu}(x)v_\lambda v_\mu + V(x)$$

It is precisely these properties that will be important later; we therefore do not need to make the connection between $G_{\lambda\mu}$, $\Gamma_{\lambda\mu\nu}$, Φ_λ and the original Lagrangian L more precise.

In expansions (1.1)we put $j = \rho\sigma\tau$, after which $\xi^j = p_-^\rho p_+^\sigma v^\tau$.

Lemma 3. The coefficients of the expansion on the right-hand sides have the following properties:

$$p_-^{[\rho\sigma\tau]} = \bar{p}_+^{[\rho\sigma\tau]} = -(-1)^{|\tau|}\bar{p}_-^{[\rho\sigma\tau]}$$

$$v^{[\rho\sigma\tau]} = \bar{v}^{[\sigma\rho\tau]} = (-1)^{|\tau|}\bar{v}^{[\rho\sigma\tau]}$$

$$X^{[\rho\sigma\tau]} = \bar{X}^{[\sigma\rho\tau]} = -(-1)^{|\tau|}X^{[\rho\sigma\tau]}$$

Indeed, because of the reversibility property of the system the variables p_-, p_+, v and X can also have t replaced by $-t$, v by $-v$, and p_- and p_+ interchanged (all simultaneously).

Corollary. We shall write τ_0 for even $|\tau|$ and τ_1 for odd $|\tau|$. If $j = \rho\sigma\tau_0$, the coefficients of v are real, p_-, p_+ and X are purely imaginary; if $j = \rho\sigma\tau_1$, the opposite is true. For $\rho = \sigma$ and $\tau = \tau_0$, τ_1 the coefficients of X and v are, respectively, equal to zero.

Theorem 1. We will assume that amongst the frequencies $\omega_\alpha(\cdot)$ there are no commensurabilities up to order $N+1$ inclusive. We will denote the variables in the normal form of the Nth approximation by q_-, q_+, w, Y and will no longer refer to the fact that they depend on N [see (1.2)].

We introduce "polar coordinates" $\psi_\alpha = \arg p_{-\alpha}$, $\Delta_\alpha = |p_{-\alpha}|$. The normal form of the Nth approximation reduces to

$$\dot{\Delta}_\alpha = \Delta_\alpha \Sigma \Delta_\alpha^{\{\rho\tau_1\}}(Y)\Delta^{2\rho}w^{\tau_1}, \qquad 1 \leqslant 2|\rho| + |\tau_1| \leqslant N-1 \tag{3.1}$$

$$\dot{w}_a = \Sigma w^{\{\rho\tau_0\}}(Y)\Delta^{2\rho}w^{\tau_0}, \qquad 2 \leqslant 2|\rho| + |\tau_0| \leqslant N \tag{3.2}$$

$$\dot{Y}_s = \Sigma Y_s^{\{\rho\tau_1\}}(Y)\Delta^{2\rho}w^{\tau_1}, \qquad 3 \leqslant 2|\rho| + |\tau_1| \leqslant N-1 \tag{3.3}$$

$$\dot{Y}_a = w_a + \Sigma Y_a^{\{\rho\tau_1\}}(Y)\Delta^{2\rho}w^{\tau_1}, \qquad 3 \leqslant 2|\rho| + |\tau_1| \leqslant N-1 \tag{3.4}$$

$$\dot{\psi}_\alpha = \omega_\alpha(Y) + \Sigma \psi_\alpha^{\{\rho\tau_0\}}(Y)\Delta^{2\rho}w^{\tau_0}, \qquad 2 \leqslant 2|\rho| + |\tau_0| \leqslant N-1 \tag{3.5}$$

Proof. The vector λ has the form $(i\omega, -i\omega, 0)$ so that $(\lambda, j) = i(\omega, \rho-\sigma)$ with $|\rho-\sigma| \leqslant$

$|\rho|+|\sigma|=|j|-|\tau|$. Relations (1.4)–(1.6) and Lemma 3 show that normalization preserves the reality property ($\bar{q}_+ \equiv q_-$) and reversibility. We will write out the coefficients that have to be present in normal form (not being reducible to zero) and re-denote them:

$$Y_k^{[\rho\rho\tau_1]} = Y_k^{\{\rho\tau_1\}}, \quad w_a^{[\rho\rho\tau_0]} = w_a^{\{\rho\tau_0\}}$$
$$q_{-\alpha}^{[\rho\rho'\tau_1]} = \Delta_\alpha^{\{\rho\tau_1\}}, \quad q_{-\alpha}^{[\rho\rho'\tau_0]} = i\psi_\alpha^{\{\rho\tau_0\}}, \quad \rho' = \rho + e_\alpha$$

Here we have also used the corollary to Lemma 3. The purpose and meaning of the transformation are obvious. The details of the second approximation are used in (3.3) and (3.4).

Remarks. 1. System (3.1)–(3.4) separates out; it describes the changes in the "slow" variables. In the transition from the $2M$th approximation to the $(2M+1)$th approximation the slow variables change in the same way, while because of (3.5) the angular variables obtain corrections to the frequency of order ϵ^{2M}.

2. In the case when the forces are not conservative, but are only functions of the coordinates, the normal form will be the same.

3. Expressions for the original variables in terms of the variables Δ, ψ, w and Y from the normal form of the Nth approximation can be taken in the form

$$x_i = y_i + \Sigma[A_i^{(\rho\sigma\tau)}(Y)\cos(\rho-\alpha,\ \psi) + B_i^{(\rho\sigma\tau)}(Y)\sin(\rho-\alpha,\ \psi)]\ \Delta^{\rho+\sigma}w^\tau \tag{3.6}$$

where, if $i=\alpha$, then $y_\alpha = \Delta_\alpha\cos\psi_\alpha$, elsewhere $\cos\psi_\alpha$ or $\sin\psi_\alpha$ with coefficient $F(X, \Delta, w)$ will not occur, in the summation $2\leqslant|j|\leqslant N$; if $i=a$ or s, then $y_a = Y_a$, $y_s = Y_s$, the coefficients $A^{(\rho\rho\sigma)}=0$ in the summation $|j|\leqslant N$ and, as above, $|j|\geqslant 2$, which is associated with the special form of the second-approximation equations. After substituting the solution of system (3.1)–(3.5) into (3.6) we obtain the Nth approximation for the solution of the exact equations in a neighbourhood of E (with an error discussed in Sec. 1).

4. If $\dim E = n-m$, then the variables w_a, X_a are not present in Theorem 1, while Eqs (3.1) and (3.4) acquire the form $\Delta_\alpha^\cdot = 0$, $Y_r^\cdot = 0$. The latter enables one to assume the constraints to be integrable, because by (3.6) X_s oscillates about constant values. These oscillations are the main manifestation of non-holonomicity (because the normal form (3.1), (3.5) is also true for holonomic systems with position forces and parameters Y_r).

5. A counter-example to remark 4 is the following system:

$$L = (x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2})/2 - (x^2+y^2)/2, \qquad z^\cdot = xy^\cdot - yx^\cdot$$

Here the equations of motion are $x^{\cdot\cdot}+x=y^{\cdot\cdot}+y=0$ plus the constraint equation, as a result of which $z^\cdot$ has the meaning of a constant square, and z increases monotonically (this being an effect of the simplest resonance $\omega_1=\omega_2$).

Theorem 2. We expand the energy integral:

$$H = \Sigma H^{(\rho\sigma\tau)}(Y) q_-^\rho q_+^\sigma w^\tau$$

Because of the reversibility (the evenness of H in $x^\cdot$) the quantities $H^{(\rho\sigma\tau_0)}$ and $iH^{(\rho\sigma\tau_1)}$ are real, so that

$$H = \Sigma[H^{(\rho\sigma\tau_0)}(Y)\cos(\rho-\sigma,\ \psi) + iH^{(\rho\sigma\tau_1)}(Y)\sin(\rho-\sigma,\ \psi)]\Delta^{\rho+\sigma}w^\tau \tag{3.7}$$

The averaged energy (a function of the slow variables only)

$$H_0 = \Sigma H_0^{[2L]} = \sum_{L=0}^{M}\ \sum_{|j|=2L} H^{(\rho\rho\tau_0)}(Y)\Delta^{2\rho}w^\tau, \quad 2M\leqslant N+1 \tag{3.8}$$

has, because of system (3.1)–(3.4), a derivative of order $2M+2$, which imposes the following restrictions on the coefficients of the system:

$$\sum_a \frac{\partial H^{(\rho\rho\tau_0)}}{\partial Y_a} Y_a^{\{\mu\nu_1\}} + \sum_r \frac{\partial H^{(\rho\rho\tau_0)}}{\partial Y_r} Y_r^{\{\mu\nu\}} + \tag{3.9}$$
$$+ \sum_\alpha 2H^{(\rho\rho\tau_0)}\rho_\alpha\Delta_\alpha^{\{\mu\nu_1\}} + \sum_a H^{(\rho,\rho,\nu_1+e_a)}(\nu_a+1)w^{\{\mu\tau_0\}} = 0$$

where $\rho+\mu$ and $\tau_0=\nu_1$ are arbitrary fixed integer vectors. The non-constant form $H_0^{[2k]}$ of lowest degree is an exact integral of the second-approximation equations.

In particular, $H^{[0]} = V(0, Y_a, Y_r)$, where there is in fact no dependence on Y_a because of (2.9). Furthermore

$$2H_0^{[2]} = \Sigma k_\alpha(Y)\Delta_\alpha^2 + \Sigma k_\alpha(Y)w_\alpha^2$$

If $H_0^{[0]} = \text{const}$, i.e. if E is a manifold of critical points of V, then $H_0^{[2]}$ is an integral for the slow variables in the first (which is trivial), second and third approximations.

Examples. We will use Theorem 1 to write out the normal form of the second approximation for $l = 2$, $m = 3$, $n = 5$:

$$\begin{aligned} &\Delta_1^{\cdot} = c_1\Delta_1 w_3, \quad \Delta_2^{\cdot} = c_2\Delta_2 w_3 \\ &w_3^{\cdot} = d_1\Delta_1^2 + d_2\Delta_2^2 + d_3 w_3^2, \quad Y_3^{\cdot} = w_3 \\ &Y_4^{\cdot} = Y_5^{\cdot} = 0, \quad \psi_1^{\cdot} = \omega_1, \quad \psi_2^{\cdot} = \omega_2 \end{aligned} \tag{3.10}$$

Suppose the coefficients are constant. A system of the form (3.10) appeared [8] in the problem of a rigid body rolling on a plane. There it is true that $d_3 = 0$. It turns out that this is a universal fact, and follows from the presence of the integral

$$2H_0^{[2]} = k_1\Delta_1^2 + k_2\Delta_2^2 + k_3 w_3^2$$

because from Theorem 2 one obtains

$$k_1 c_1 + k_3 d_1 = k_2 c_2 + k_3 d_2 = k_3 d_3 = 0$$

The analysis of system (4.1) in [8] carries over to a wide class of systems for which $c_1, c_2 < 0$. A similar analysis is possible for other cases. A system of the form (3.10) with non-constant coefficients appears in the problem of the rolling of a plane plate on a rough surface in a gravitational field. It is extremely complicated to reduce it to normal form in the higher approximations (see Sec. 4).

4. THE PROBLEM OF A ROD ROLLING ON AN INCLINED CYLINDER

Suppose that in a cylindrical system of coordinates $P(r, \varphi, z)$ is the point of contact of the rod and the cylinder, S is its centre of mass, $PS = s\mathbf{e}$, where $\mathbf{e}$ is the direction vector of the rod, and θ is the angle between $\mathbf{e}_\varphi$ and $\mathbf{e}$. Then without taking account of the constraints

$$T = \tfrac{1}{2}M([r\varphi^{\cdot} + (s\cos\theta)^{\cdot}]^2 + [z^{\cdot} + (s\sin\theta)^{\cdot}]^2 + [s\cos\theta\varphi^{\cdot}]^2) + \tfrac{1}{2}I(\theta^{\cdot 2} + \varphi^{\cdot 2}\cos^2\theta)$$

where I is the central moment of inertia of the rod and M is its mass. The constraint equations are

$$dz + ds\cdot\sin\theta = 0, \quad rd\varphi + ds\cdot\cos\theta = 0 \tag{4.1}$$

For inertial motion we have the Chaplygin system with independent variables s and θ. Moreover, the change of variable $dt = \cos\theta(I + Ms^2)^{1/2}d\tau$ reduces the equations of inertial motion to Lagrangian form.

The potential energy with an appropriate choice of φ has the form

$$V = Mg[\sin\alpha(z + s\sin\theta) + \cos\alpha(r\cos\varphi - s\cos\theta\sin\varphi)]$$

where α is the angle between the axis of the cylinder and the horizontal plane. One can assume that $M = g = r = 1$.

The coordinates prescribed by Lemma 1 are introduced by the formulas

$$\Phi = \varphi + s/\cos\theta, \quad Q = z + s\sin\theta$$

because the equilibrium manifold is $E = \{s = 0\}$.

Corollary to Theorems 1 and 2. The normal form of the second approximation equations (3.1) and (3.2) for $l = 1$, $m = 2$ has the form ($\Delta \equiv \Delta_1$, $w = w_2$)

$$\Delta^{\cdot} = c(Y)\Delta w, \quad w^{\cdot} = d_1(Y)\Delta^2 + d_2(Y)w^2$$

In the third approximation to equation $\psi^{\cdot} = \omega$ (see Eq. (3.5)) corrections quadratic in Δ and w are added. In the fourth approximation on the right-hand sides of $\Delta^{\cdot}$ and $w^{\cdot}$ there appear terms of the fourth degree in Δ and

w (which we shall not make more precise or write out), and, most importantly, a transgression in the sense of [9] can begin: the non-trivial evolution of the dependent coordinates

$$Y_s^{\cdot} = w(E_{s1}(Y)\Delta^2 + E_{s2}(Y)w^2), \quad s = 3, 4, \ldots$$

(similar terms are also added to $Y_2^{\cdot} = w$).

The energy integral has the form

$$H = V(Y_s) + \tfrac{1}{2}[k_1(Y)\Delta^2 + k_2(Y)w^2] + \ldots$$

and relations (3.9) give

$$\sum \frac{\partial V}{\partial Y_s} E_{s\lambda} + \frac{\partial k_\lambda}{\partial Y_2} + 2k_2 d_\lambda - 2(\lambda - 2)k_1 c_1 = 0, \quad \lambda = 1, 2$$

Results of calculations. In the problem under consideration

$$\Delta^2 \approx s^2 + s'^2/\omega^2, \quad w \approx \theta'/\cos\theta + s'F/\omega^2, \quad Y_3 \approx \Phi, \quad Y_4 \approx Z$$

$$\omega^2(\Phi) = \cos\alpha\cos\Phi, \quad F = \cos\theta\sin\alpha + \cos\alpha\sin\Phi\sin\theta$$

$$c_1 = \sin\theta, \quad d_1 = d_2 = 0, \quad k_1 = \cos^2\theta\cos\alpha\cos\Phi, \quad k_2 = 1$$

$$E_{31} = -\frac{3}{4}\frac{\sin\alpha}{\cos^3\theta}, \quad E_{41} = -\frac{3}{4}\frac{\cos\alpha\sin\Phi}{\cos^2\theta}, \quad E_{32} = E_{42} = 0$$

The process of the motion can be qualitatively represented as oscillations in s, φ, z with amplitude of order ϵ about an equilibrium position with coordinates Ψ, Q combined with the slow rotation of the rod. (In the first approximation the latter effect is not present.) In a time of order $1/\epsilon$ the angle θ changes by a finite amount, and the equilibrium position about which the oscillation occurs is displaced by an amount of order ϵ^2 (transgression). The displacement is along the curve

$$Q = Q_0 + \operatorname{ctg}\alpha[\cos(\Phi - \Phi_0) - 1]$$

The amplitude of the oscillations is of order Δ. In the second approximation $\Delta\cos\theta = \Delta_0\cos\theta_0$, so that depending on the direction of the change in θ the quantity Δ decreases or increases, remaining a quantity of order ϵ.

5. CONCLUSION

The concept of weak non-holonomicity proposed in [9] and used in [10, 17, 18] consisted of considering a non-holonomic system depending on a small parameter such that when the latter was zero one obtained a family of Hamiltonian systems. Perturbations of such a family then appear in the investigation of a single non-holonomic system.

The main conclusions are as follows (see also [19]).

If the dimension of the equilibrium manifold is equal to the number of constraints, then in dynamics with independent frequencies the constraint equations are "integrable on average", i.e. in suitably defined coordinates the motion is nearly confined to coordinate planes, and the deviation from the latter is of the second order of smallness and has an oscillatory nature.

If the dimension of the equilibrium manifold is greater than the number of constraints, in the second approximation a trivial displacement along it occurs with velocity of the first order of smallness, while in the fourth approximation one can catch a previously unnoticed effect of additional evolution along the equilibrium manifold with a velocity of the third order of smallness, so that it is no longer appropriate to talk of the "average integrability".

The examples given above show that near the equilibrium manifold of a non-holonomic system all the richness of Hamiltonian mechanical phenomena can appear. In addition, from the point of view of normal forms when there is no commensurability between the frequencies the behaviour of non-holonomic systems near an equilibrium manifold is very uniform and is mainly governed by its dimension. The exception is problems in which, for one reason or another (for example, some

symmetry), coefficients of a normal form vanish, as in the problem of the oscillation of a body on a plane in which the point of contact does not evolve. The existence of an energy integral enables us to draw definite conclusions about the coefficients of the higher approximations when the lower coefficients are known.

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